Spin squeezing inequalities and entanglement of N qubit states

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We derive spin squeezing inequalities that generalize the concept of the spin squeezing parameter and provide necessary and sufficient conditions for genuine 2-, or 3- qubit entanglement for symmetric states, and sufficient condition for general N-qubit states. Our inequalities have a clear physical interpretation as entanglement witnesses, can be easy measured, and are given by complex, but *elementary* expressions.

Recently, the area of quantum correlated systems of atoms or ions, and in particular *mesoscopic ionic* and *macroscopic atomic ensembles* [1] has been developing very rapidly. Spin squeezing of, say few ions to 10^7 atoms is nowadays routinely achieved in such systems. The standard tool to detect the generated forms of multipartite entanglement [2, 3] provides the, so called, spin squeezing parameter ξ^2 introduced in Ref. [4]. The spin squeezing parameter is particularly appreciated by experimentalists for the following reasons: i) it has a clear physical meaning, ii) it can be relatively easy measured, iii) it is defined by a simple operation expression, iv) it provides a figure of merit for atomic clocks. Moreover, as shown in [5, 6], ξ^2 is directly connected to entanglement in atomic ensembles, providing a sufficient entanglement condition. However, one should stress that no further investigations to relate ξ^2 to other concepts of quantum information have been carried out so far.

In this Letter we generalize and connect the concept of spin squeezing parameters to the theory of entanglement witnesses [7], i. e. such observables $\mathcal{W}$ that have non-negative averages for all separable states and there exists an entangled state ϱ such that $\mathrm{tr}(\varrho\mathcal{W}) < 0$. In order to derive the generalized spin squeezing inequalities, we express state averages of the appropriate entanglement witnesses in terms of the macroscopic spin operators:

$$J^i = \sum_{a=1}^{N} \tfrac{1}{2}\sigma_a^i \;\; i = 1, 2, 3 \;, \tag{1}$$

(σ^i denote Pauli matrices and indices $a, b, c \ldots$ enumerate the particles of the ensemble). We recall [4] that a state of a spin-J system is called spin squeezed if there exists a direction $\mathbf{n}$, orthogonal to the mean spin $\langle \mathbf{J} \rangle$, such that:

$$\xi^2 = 2\langle\Delta J_{\mathbf{n}}^2\rangle/J < 1, \tag{2}$$

where $J_{\mathbf{n}} = \mathbf{n} \cdot \mathbf{J}$.

In the proposed approach we begin with considering symmetric states of N qubits first, i.e. states ϱ supported on the symmetrized product of individual qubit spaces $\mathcal{H}_s = \mathrm{Sym}(\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2)$ (Sym denotes symmetrization). We then use the fact that for symmetric states of 2, and 3 qubits separability is equivalent to positivity of the, so called, partial transpose of a state [8]

(PPT condition [9]). From that we derive the complete families of generalized spin squeezing inequalities, which provide necessary and sufficient conditions for genuine 2-, or 3- qubit entanglement for symmetric states; at the same time they provide a sufficient condition for general states of N qubits [10]. Our results imply that spin squeezing leads to the genuine 2-qubit entanglement (i.e. the corresponding reduced two-qubit density matrices are entangled) [6]. For symmetric states the converse is also true: 2-qubit entangled states show a specific type of spin squeezing. In addition, we obtain somewhat simpler necessary conditions for the 3-qubit case, that lead to entanglement not implied by the standard spin squeezing. The proposed novel inequalities, similarly as the squeezing parameter, i) have a clear physical meaning in terms of generalized squeezing and entanglement conditions, ii) can be relatively easy measured, and iii) are given by complex, but *elementary* expressions.

The simplest form of entanglement that a multiqubit state ϱ can possess is a 2-qubit entanglement: ϱ is 2-qubit entangled if for some qubits a and b the reduced density matrix

$$\varrho_{ab} = \mathrm{tr}_{1..\hat{a}..\hat{b}..N}\varrho \tag{3}$$

is entangled (the hats over indices mean that those indices are omitted). Let us first consider symmetric states. Then all the reductions ϱ_{ab} are of the same form and act in a symmetric subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$ - the space of qubits a and b. The PPT criterion [9] implies that ϱ_{ab} is entangled iff there exists a vector ψ such that

$$\mathrm{tr}_{ab}\left(\varrho_{ab}|\psi\rangle\langle\psi|^{T_1}\right) < 0, \tag{4}$$

where transpose is defined w.r.t. the standard basis $|0\rangle, |1\rangle$. As ψ we can take any eigenvector of $\varrho_{ab}^{T_1}$, corresponding to a negative eigenvalue.

From the explicit form of $\varrho_{ab}^{T_1}$ we deduce that $|\psi\rangle$ can be parametrized as follows [11]: $|\psi\rangle = \eta|00\rangle + \beta|01\rangle + \beta^*|10\rangle + \gamma|11\rangle$, with $\alpha, \gamma \in \mathbb{R}$. Hence the coefficients of $|\psi\rangle$ form a hermitean matrix: $[\psi_{CD}]_{C,D=0,1}$. We can diagonalize it: $\psi_{CD} = \tilde{U}_{AC}^*\Delta_{AB}\tilde{U}_{BD}$, where $\Delta = \mathrm{diag}(\sin\frac{\alpha}{2}, \pm\cos\frac{\alpha}{2})$, $-\pi \leq \alpha \leq \pi$, $\tilde{U} \in SU(2)$, and then define $U = \sum_{C,D=0,1} \tilde{U}_{CD}|D\rangle\langle C|$ to finally obtain

the following parametrization:

$$|\psi\rangle = U^* \otimes U|\psi_0\rangle, \quad |\psi_0\rangle = \sin\tfrac{\alpha}{2}|00\rangle + \cos\tfrac{\alpha}{2}|11\rangle, \quad (5)$$

(we have fixed the overall phase). Substituting (5) into (4) leads to the condition:

$$\mathrm{tr}_{ab}\big(\varrho_{ab}U \otimes U|\psi_0\rangle\langle\psi_0|^{T_1}U^\dagger \otimes U^\dagger\big) < 0. \quad (6)$$

Note, that $|\psi_0\rangle\langle\psi_0|^{T_1}$ can be decomposed into Pauli matrices:

$$
\begin{aligned}
|\psi_0\rangle\langle\psi_0|^{T_1} &= \tfrac{1}{4}\sin^2\tfrac{\alpha}{2}\big(\mathbf{1}+\sigma^z\big)\otimes\big(\mathbf{1}+\sigma^z\big)\\
&+\tfrac{1}{4}\cos^2\tfrac{\alpha}{2}\big(\mathbf{1}-\sigma^z\big)\otimes\big(\mathbf{1}-\sigma^z\big)\\
&+\tfrac{1}{4}\sin\alpha\big(\sigma^x\otimes\sigma^x+\sigma^y\otimes\sigma^y\big),
\end{aligned}
\quad (7)
$$

and the adjoint action of $SU(2)$ in (6) induces a $SO(3)$ rotation R of σ^i: $U\sigma^i U^\dagger = R^i_{\ j}\sigma^j$ (here and throughout we sum over repeated indices). We will denote the axes of the rotated frame by $\mathbf{k}, \mathbf{l}, \mathbf{n}$.

Using (7) we can express the inequality (6) through the rotated total spin operators (1). We first observe that $\mathrm{tr}_{ab}\big(\varrho_{ab}|\psi\rangle\langle\psi|^{T_1}\big) = \mathrm{tr}\big(\varrho|\psi_{ab}\rangle\langle\psi_{ab}|^{T_1}\big)$, where $|\psi_{ab}\rangle$ is the natural embedding of $|\psi\rangle$ into $\mathcal{H}$. Since all ϱ_{ab} are of the same form, we can sum (6) over all pairs of qubits: $\sum_{\langle ab\rangle} = \sum_{a=1}^{N-1}\sum_{b=a+1}^{N}$ and use the identity: $\sum_{\langle ab\rangle}\sigma_a^i \otimes \sigma_b^i = 2(J^i)^2 - N/2$ to obtain the following inequality [12]:

$$
\begin{aligned}
&\sin\alpha\big(\tfrac{N^2}{4}-\langle J_{\mathbf{n}}^2\rangle\big) - (N-1)\cos\alpha\langle J_{\mathbf{n}}\rangle\\
&+\langle J_{\mathbf{n}}^2\rangle + \tfrac{N(N-2)}{4} < 0,
\end{aligned}
\quad (8)
$$

where the averages are taken w.r.t. ϱ.

Let us now fix the direction $\mathbf{n}$ and minimize the l.h.s. of the inequality (8) w.r.t. α. We find that the inequality (8) is satisfied if and only if:

$$\langle J_{\mathbf{n}}^2\rangle + \tfrac{N(N-2)}{4} < \sqrt{\big(\tfrac{N^2}{4}-\langle J_{\mathbf{n}}^2\rangle\big)^2 + (N-1)^2\langle J_{\mathbf{n}}\rangle^2}. \quad (9)$$

For a general, i.e. not necessarily symmetric, state ϱ we can still test entanglement of all the bipartite reductions ϱ_{ab} with the *same* vector (5). The sum $2\mathrm{tr}\big(\varrho\sum_{\langle ab\rangle}|\psi_{ab}\rangle\langle\psi_{ab}|^{T_1}\big)$ is then not greater than the l.h.s of (8) due to [12] and we finally obtain from (9):

Criterion for bipartite entanglement. *If there exists a direction* $\mathbf{n}$ *such that the following inequality holds:*

$$\frac{4\langle\Delta J_{\mathbf{n}}^2\rangle}{N} < 1 - \frac{4\langle J_{\mathbf{n}}\rangle^2}{N^2} \quad (10)$$

then the state ϱ *possesses bipartite entanglement. For symmetric states the above condition is both necessary and sufficient.*

To relate the above criterion to the standard spin squeezing condition (2) for spin-J states, note that if (2) is satisfied for some direction $\mathbf{n}$ then so is (10) as

$\langle J_{\mathbf{n}}\rangle = 0$ and $J \leq N/2$. Hence, spin squeezed states possess 2-qubit entanglement (for symmetric states this was proven in Ref. [6]). For symmetric states, for which $J = N/4$, the (modified) converse also holds: condition (10) implies existence of a spin component $J_{\mathbf{n}}$ such that $\langle\Delta J_{\mathbf{n}}^2\rangle < N/4$. This differs from the standard definition of spin squeezing (2) in that the direction $\mathbf{n}$ need not be orthogonal to $\langle\mathbf{J}\rangle$. Nevertheless, we also call such states spin squeezed.

Let us now consider the case when ϱ possesses genuine 3-qubit entanglement, i.e. for some triple of qubits abc, the reduced density matrix

$$\varrho_{abc} = \mathrm{tr}_{1..\hat{a}..\hat{b}..\hat{c}..N}\varrho \quad (11)$$

is 3-party entangled. If we again consider symmetric states first, then PPT criterion is still necessary and sufficient for separability, since $\mathrm{Sym}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$ is a subspace of $\mathbb{C}^2 \otimes \mathbb{C}^3$. Thus we can proceed as before.

A vector $|\psi\rangle$, corresponding to any negative eigenvalue of $\varrho_{abc}^{T_1}$ must be necessarily a 3-party entangled vector from $\mathbb{C}^2 \otimes \mathrm{Sym}(\mathbb{C}^2 \otimes \mathbb{C}^2)$. The parametrization of such vectors was found in Ref. [13]; there are two families:

$$|\psi\rangle = A \otimes B \otimes B|GHZ\rangle \quad (12)$$
$$|\psi\rangle = A \otimes U \otimes U|W\rangle, \quad (13)$$

where matrices $A, B \in SL(2,\mathbb{C})$, $U \in SU(2)$, and $|GHZ\rangle = (1/\sqrt{2})(|000\rangle + |111\rangle)$, $|W\rangle = (1/\sqrt{3})(|001\rangle + |010\rangle + |100\rangle)$. The action of $SL(2,\mathbb{C})$ on the Pauli matrices in the decomposition of $|\psi\rangle\langle\psi|^{T_1}$ now induces restricted, i.e. orientation and time-orientation preserving, Lorenz transformations:

$$A^*\sigma^\mu A^T = \Lambda^\mu_{\ \nu}\sigma^\nu, \quad B\sigma^\mu B^\dagger = L^\mu_{\ \nu}\sigma^\nu, \quad \sigma^0 = \mathbf{1}, \quad (14)$$

(Greek indices run through $0\ldots4$). Hence we obtain the following inequality, analogous to (6):

$$\mathrm{tr}_{abc}\big(\varrho_{abc}|\psi\rangle\langle\psi|^{T_1}\big) = \tfrac{1}{8}K_{\alpha\beta\gamma}\langle\sigma_a^\alpha \otimes \sigma_b^\beta \otimes \sigma_c^\gamma\rangle < 0, \quad (15)$$

where

$$
\begin{aligned}
K_{\alpha\beta\gamma}(\Lambda, L, L) &= \Lambda^0_{\ \alpha}L^0_{\ \beta}L^0_{\ \gamma} + \Lambda^0_{\ \alpha}L^3_{\ \beta}L^3_{\ \gamma} + \Lambda^1_{\ \alpha}L^1_{\ \beta}L^1_{\ \gamma}\\
&+2\Lambda^3_{\ \alpha}L^0_{\ (\beta}L^3_{\ \gamma)} - \Lambda^1_{\ \alpha}L^2_{\ \beta}L^2_{\ \gamma} + 2\Lambda^2_{\ \alpha}L^1_{\ (\beta}L^2_{\ \gamma)},
\end{aligned}
\quad (16)
$$

for the GHZ family (12), or

$$
\begin{aligned}
K_{\alpha\beta\gamma}(\Lambda, R, R) &= \tfrac{1}{3}\Big\{3\Lambda^0_{\ \alpha}R^0_{\ \beta}R^0_{\ \gamma} - 3\Lambda^3_{\ \alpha}R^3_{\ \beta}R^3_{\ \gamma}\\
&+2\Lambda^0_{\ \alpha}R^0_{\ (\beta}R^3_{\ \gamma)} + \Lambda^3_{\ \alpha}R^0_{\ \beta}R^0_{\ \gamma} - \Lambda^0_{\ \alpha}R^3_{\ \beta}R^3_{\ \gamma}\\
&-2\Lambda^3_{\ \alpha}R^0_{\ (\beta}R^3_{\ \gamma)} + 4\Lambda^1_{\ \alpha}R^0_{\ (\beta}R^1_{\ \gamma)} + 4\Lambda^1_{\ \alpha}R^1_{\ (\beta}R^3_{\ \gamma)}\\
&-4\Lambda^2_{\ \alpha}R^0_{\ (\beta}R^2_{\ \gamma)} - 4\Lambda^2_{\ \alpha}R^2_{\ (\beta}R^3_{\ \gamma)}\Big\}
\end{aligned}
\quad (17)
$$

for the W family (13). Here $R^\mu_{\ \nu}$ is the four-dimensional embedding of the rotation generated by U from (13) and round brackets denote symmetrization.

In order to express the inequality (15) through ϱ-averages of the spin operators (1), we introduce an artificial time-component $J^0 = (N/2)\mathbf{1}$. The operators $J^\mu = (J^0, J^i)$ do not constitute relativistic generalization of the operators J^i and we introduce them just for notational reasons. Since ϱ_{abc} is symmetric, the indices $\alpha\beta\gamma$ in (15) can be symmetrized. Then after summing (15) over all triples of qubits $\sum_{\langle abc\rangle} = \sum_{a=1}^{N-2}\sum_{b=a+1}^{N-1}\sum_{c=b+1}^{N}$, we can use the identity:

$$3\sum_{\langle abc\rangle} \sigma_a^{(\alpha}\otimes\sigma_b^{\beta}\otimes\sigma_c^{\gamma)} = 4J^{(\alpha}J^\beta J^{\gamma)} - 6f^{(\alpha\beta}{}_\mu J^\gamma J^{\mu)}$$
$$+2f^{(\alpha\beta}{}_\mu f^{\gamma\mu)}{}_\nu J^\nu, \qquad (18)$$

where $f^{0\alpha}{}_\beta = f^{\alpha0}{}_\beta = \delta^\alpha{}_\beta$, $f^{ij}{}_\alpha = i\sum_l \epsilon^{ijl}\delta^l{}_\alpha + \delta^{ij}\delta^0{}_\alpha$, to finally obtain:

Criterion for tripartite entanglement. *A symmetric state ϱ possesses a genuine tripartite entanglement iff there exist two restricted Lorenz transformations Λ, L, or a restricted Lorenz transformation Λ and a rotation R, such that:*

$$K_{(\alpha\beta\gamma)}\Big\{2\langle J^\alpha J^\beta J^\gamma\rangle - 3f^{\alpha\beta}{}_\mu\langle J^{(\gamma}J^{\mu)}\rangle + f^{\alpha\beta}{}_\mu f^{(\gamma\mu)}{}_\nu\langle J^\nu\rangle\Big\} < 0 \qquad (19)$$

holds, with $K_{\alpha\beta\gamma}$ given by (16), or by (17) respectively.

The above criterion serves also as a sufficient condition for tripartite entanglement for a general state ϱ, with the modification that $K(\Lambda, L, L)$ or $K(\Lambda, R, R)$ in (19) have to be substituted with $1/3[K(\Lambda,L,L) + K(L,\Lambda,L) + K(L,L,\Lambda)]$ or $1/3[K(\Lambda,R,R) + K(R,\Lambda,R) + K(R,R,\Lambda)]$ respectively to achieve the index symmetrization.

The search for matrices Λ, L can be difficult due to non-compactness of the restricted Lorenz group. It is therefore desirable to develop some simpler conditions as well. For mesoscopic systems with not too large N we may do so, using some specific witnesses that detect genuine GHZ-type, or genuine W-type entanglement, found in Ref. [14]:

$$\mathcal{W}_{GHZ} = \tfrac{3}{4}\mathbf{1} - |GHZ\rangle\langle GHZ| \qquad (20)$$
$$\mathcal{W}_{W_1} = \tfrac{2}{3}\mathbf{1} - |W\rangle\langle W| \qquad (21)$$
$$\mathcal{W}_{W_2} = \tfrac{1}{2}\mathbf{1} - |GHZ\rangle\langle GHZ|, \qquad (22)$$

where, in order to be more general, we may now define the vectors $|GHZ\rangle$ and $|W\rangle$ in an arbitrary frame $\mathbf{k}, \mathbf{l}, \mathbf{n}$, rotated w.r.t. the original one. The witnesses $\mathcal{W}_{GHZ}$ detects states of GHZ class which are neither of the W class, nor biseparable. Finally, the witnesses $\mathcal{W}_{W_1}$ and $\mathcal{W}_{W_2}$ detect states of GHZ- or W-class, which are not biseparable [14]. Proceeding as before and using the same witnesses (20)-(22) for all tripartite reductions ϱ_{abc} of a general state ϱ, we get necessary conditions for:

GHZ-type entanglement. *If for a state ϱ there exist orthogonal directions $\mathbf{k}, \mathbf{l}, \mathbf{n}$ such that the following inequality is fulfilled*

$$-\tfrac{1}{3}\langle J_{\mathbf{k}}^3\rangle + \langle J_{\mathbf{l}}J_{\mathbf{k}}J_{\mathbf{l}}\rangle - \tfrac{N-2}{2}\langle J_{\mathbf{n}}^2\rangle + \tfrac{1}{3}\langle J_{\mathbf{k}}\rangle$$
$$+ \tfrac{N(N-1)(5N-2)}{24} < 0, \qquad (23)$$

then the state ϱ possesses a genuine GHZ-type entanglement.

GHZ- or W-type entanglement. *If for a state ϱ there exist orthogonal directions $\mathbf{k}, \mathbf{l}, \mathbf{n}$ such that one of the following inequalities is fulfilled*

$$\langle J_{\mathbf{n}}^3\rangle - 2\langle J_{\mathbf{l}}J_{\mathbf{n}}J_{\mathbf{l}}\rangle - 2\langle J_{\mathbf{k}}J_{\mathbf{n}}J_{\mathbf{k}}\rangle$$
$$-\tfrac{N-2}{2}\big(2\langle J_{\mathbf{k}}^2\rangle + 2\langle J_{\mathbf{l}}^2\rangle - \langle J_{\mathbf{n}}^2\rangle\big) - \tfrac{N^2-4N+8}{4}\langle J_{\mathbf{n}}\rangle$$
$$+ \tfrac{N(N-2)(13N-4)}{24} < 0 \qquad (24)$$
$$-\tfrac{1}{3}\langle J_{\mathbf{k}}^3\rangle + \langle J_{\mathbf{l}}J_{\mathbf{k}}J_{\mathbf{l}}\rangle - \tfrac{N-2}{2}\langle J_{\mathbf{n}}^2\rangle + \tfrac{1}{3}\langle J_{\mathbf{k}}\rangle$$
$$+ \tfrac{N^2(N-2)}{8} < 0, \qquad (25)$$

then the state ϱ possesses a genuine 3-qubit (GHZ- or W-type) entanglement.

The above spin squeezing criteria (10),(19), (23)-(25) constitute the main result of this Letter. The inequalities (23) and (25) can be further simplified if we choose the directions $\mathbf{k}, \mathbf{n}$ such that $\langle J_{\mathbf{k}}\rangle = \langle J_{\mathbf{n}}\rangle = 0$. Let us further assume that: $\langle J_{\mathbf{k}}^2\rangle \geq N/4$, $\langle J_{\mathbf{n}}^2\rangle \geq N/4$, so that there is no spin-squeezing in the sense of the definition (2). Then from criteria (23), (25) it follows, that if:

$$-\tfrac{1}{3}\langle J_{\mathbf{k}}^3\rangle + \langle J_{\mathbf{l}}J_{\mathbf{k}}J_{\mathbf{l}}\rangle + \tfrac{N(5N^2-10N+8)}{24} < 0 \qquad (26)$$

or:

$$-\tfrac{1}{3}\langle J_{\mathbf{k}}^3\rangle + \langle J_{\mathbf{l}}J_{\mathbf{k}}J_{\mathbf{l}}\rangle + \tfrac{N(N-1)(N-2)}{8} < 0 \qquad (27)$$

holds, then the state ϱ possesses a genuine GHZ or 3-qubit entanglement respectively. Thus, in this specific situation, the inequalities (26) and (27) detect a different type of entanglement than that implied by the standard spin squeezing [5, 6].

Generalization of the above procedure to study the entanglement between more qubits is straightforward - one uses inequalities of the type $\mathrm{tr}(\varrho\mathcal{W}) < 0$ with appropriable witnesses $\mathcal{W}$. However, for the case of four or more qubits the PPT criterion is no longer sufficient and only necessary conditions of the type (23)-(25) can be obtained.

Let us conclude with a general remark concerning full (i.e. N-qubit) separability of a symmetric state and a connection to the method of Ref. [15]. Every symmetric state ϱ of N-qubits admits an analog of Glauber-Sudarshan P-representation [16, 17]:

$$\varrho = \int_{\mathbb{S}^2} d\Omega P(\theta,\phi) |\theta,\phi\rangle\langle\theta,\phi| \otimes \cdots \otimes |\theta,\phi\rangle\langle\theta,\phi|, \quad (28)$$

where $d\Omega = \sin\theta\, d\theta d\phi$ is the volume element on the Bloch sphere, and $|\theta,\phi\rangle = \cos(\tfrac{\theta}{2})|0\rangle + e^{i\phi}\sin(\tfrac{\theta}{2})|1\rangle$ is a spin

coherent state of a single qubit. Note that every qubit is representable in this form. The representation (28) is not unique, as in the decomposition of $P(\theta, \phi)$ over spherical harmonics Y_{lm}, ϱ determines only terms with $l \leq N$, and hence $P(\theta, \phi)$ can be chosen to be a polynomial in the Cartesian coordinates on the sphere. Now the following fact holds [18, 19]:

A symmetric state ϱ is fully separable iff there exists a representation (28) where $P(\theta, \phi)d\Omega$ is an element of a probabilistic measure on $\mathbb{S}^2$

Proof. Implication $\Leftarrow$ is obvious as the integral in (28) is a norm limit of separable states. To prove the implication $\Rightarrow$, observe that if ϱ is separable, then it can be decomposed as $\varrho = \sum_k p_k |\theta_k, \phi_k\rangle\langle\theta_k, \phi_k| \otimes \cdots \otimes |\theta_k, \phi_k\rangle\langle\theta_k, \phi_k|$, $p_k \geq 0$, $\sum p_k = 1$, as vectors of the form $|\theta_k, \phi_k\rangle\langle\theta_k, \phi_k| \otimes \cdots \otimes |\theta_k, \phi_k\rangle\langle\theta_k, \phi_k|$ are the only symmetric product vectors. We define then $P(\theta, \phi) = \sum_k p_k \delta(\cos\theta - \cos\theta_k)\delta(\phi - \phi_k)$; the expansion of δ's over Y_{lm} can be truncated at $l = N$ $\square$

We observe that if $\mathcal{W}$ is an entanglement witness, then:

$$\operatorname{tr}(\varrho\mathcal{W}) = \int d\Omega P(\theta, \phi)\, w(\theta, \phi) \qquad (29)$$

where $w(\theta, \phi) = \langle(\theta, \phi)^{\otimes N}|\mathcal{W}|(\theta, \phi)^{\otimes N}\rangle$ is a positive semidefinite polynomial of the Nth order in the Cartesian coordinates. Hence, the criteria (8) and (19), with the reversed inequality signs, can be interpreted as necessary and sufficient conditions for $P(\theta, \phi)d\Omega$ to be an element of a probabilistic measure for $N = 2, 3$ respectively.

The above fact establishes an interesting link between separability of symmetric states and the problem of description of classical states of a $1D$ harmonic oscillator [15, 20]. In the latter problem, classical states are in one-to-one correspondence with probabilistic measures on $\mathbb{R}^2$. We have proved in [15] that, among some specific subclass of states, the classical ones are detected by observables, arising from positive semidefinite (psd) polynomials which are sums of squares of other polynomials.

Summarizing, we have introduced a method of deriving generalized spin squeezing inequalities, that characterize genuine N-qubit entanglement. The results of the paper provide connection of spin squeezing to entanglement witnesses, and an alternative physical meaning to spin squeezing as qualitative and quantitative characterization of the N-qubit entanglement. The inequalities can be directly measured and provide novel entanglement detection tools for macroscopic atomic ensembles.

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[10] Note, that macroscopic atomic ensembles may be prepared in symmetric subspace, although ultimately individual spontaneous emission acts take the system out of this subspace. Nevertheless, the symmetric component might remain significant for long times.

[11] For a general bipartite symmetric state we have:

$$\varrho^{T_1} = \begin{bmatrix} \epsilon_0 & \delta & \delta^* & \tau \\ \delta^* & \epsilon_1 & \omega^* & \pi^* \\ \delta & \omega & \epsilon_1 & \pi \\ \tau & \pi & \pi^* & \epsilon_2 \end{bmatrix}$$

with $\epsilon_0, \epsilon_1, \epsilon_2$ and τ real. Then it is easy to chek that vectors of the type $\eta|00\rangle + \beta|01\rangle + \beta^*|10\rangle + \gamma|11\rangle$ are preserved by ϱ^{T_1} and since they have three independent parameters (we take them to be normalized) it is possible to find a solution of the eigenvalue equation.

[12] Note that generally $\langle J_{\mathbf{k}}^2\rangle + \langle J_{\mathbf{l}}^2\rangle + \langle J_{\mathbf{n}}^2\rangle \leq N(N+2)/4$ and for symmetric states the equality holds.

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